

# On Codes based on $BCK$ -algebras

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**Abstract.** In this paper, we present some new connections between  $BCK$ -algebras and binary block codes.

**Keywords:**  $BCI/BCK$ -algebras; Binary block codes; Partially ordered set.

**AMS Classification.** 06F35, 94B60.

## 1 Introduction

$BCI/BCK$ -algebras were first introduced in mathematics in 1966 by Y. Imai and K. Iseki, through the paper [Im, Is; 66], as a generalization of the concept of set-theoretic difference and propositional calculi. One of the recent applications of  $BCK$ -algebras was given in the Coding Theory (see [Fl; 14], [Ju, So; 11]).

## 2 Preliminaries

**Definition 2.1.** An algebra  $(X, *, \theta)$  of type  $(2, 0)$  is called a  $BCI$ -algebra if the following conditions are fulfilled:

- $BCI$ -1  $((x * y) * (x * z)) * (z * y) = \theta$
- $BCI$ -2  $(x * (x * y)) * y = \theta$
- $BCI$ -3  $x * x = \theta$
- $BCI$ -4  $x * y = \theta$  and  $y * x = \theta$  imply  $x = y$

If a *BCI*-algebra  $X$  satisfies the following identity:

- *BCK*-5  $\theta * x = \theta$

then  $X$  is called a *BCK-algebra* [Me, Ju; 94].

The partial order relation on a *BCI/BCK*-algebra is defined such that  $x \leq y$  if and only if  $x * y = \theta$ .

A *BCI/BCK*-algebra  $X$  is called *commutative* if  $x * (x * y) = y * (y * x)$ , for all  $x, y \in X$  and *implicative* if  $x * (y * x) = x$ , for all  $x, y \in X$ .

If  $(X, *, \theta)$  and  $(Y, \circ, \theta)$  are two *BCI/BCK*-algebras, a map  $f : X \rightarrow Y$  with the property  $f(x * y) = f(x) \circ f(y)$ , for all  $x, y \in X$ , is called a *BCI/BCK-algebras morphism*. If  $f$  is a bijective map, then  $f$  is an *isomorphism* of *BCI/BCK*-algebras [Me, Ju; 94].

Hereafter in this paper,  $X$  always denotes a finite *BCI/BCK*-algebra.

In the following, we will use some notations and results given in the paper [Ju, So; 11].

**Definition 2.2.** A mapping  $\tilde{A} : A \rightarrow X$  is called a *BCK*-function on  $A$ , which  $A$  and  $X$  is a nonempty set and a *BCK*-algebra, respectively.

**Definition 2.3.** A cut function of  $\tilde{A}$ , for  $q \in X$ , is defined to be a mapping

$$\tilde{A}_q : A \rightarrow \{0, 1\}$$

such that

$$(\forall x \in A)(\tilde{A}_q(x) = 1 \Leftrightarrow q * \tilde{A}(x) = \theta)$$

**Definition 2.4.** Let  $A = \{1, 2, \dots, n\}$  and let  $X$  be a *BCK*-algebra. In [Ju, So; 11], to each *BCK*-function  $\tilde{A} : A \rightarrow X$  can be associated a binary block-code of length  $n$ . A codeword in a binary block-code  $V$  is  $v_x = x_1 x_2 \dots x_n$  such that  $x_i = x_j \Leftrightarrow \tilde{A}_x(i) = j$  for  $i \in A$  and  $j \in \{0, 1\}$ .

Let  $v_x = x_1 x_2 \dots x_n$  and  $v_y = y_1 y_2 \dots y_n$  be two codewords belonging to a binary block-code  $V$ . Define an order relation  $\leq_c$  on the set of codewords belonging to a binary block-code  $V$  as follows [Ju, So; 11]:

$$v_x \leq_c v_y \Leftrightarrow y_i \leq x_i \text{ for } i = 1, 2, \dots, n.$$

### 3 Main results

**Definition 3.1.** Let  $(S, \leq)$  be a partially ordered set. For  $q \in S$ , we define a mapping

$$S_q : S \rightarrow \{0, 1\}$$

such that

$$(\forall b \in S)(S_q(b) = 1 \Leftrightarrow q \leq b).$$

A codeword  $v_x = x_1x_2 \cdots x_n$  of a binary block-code  $V$  is determined as follow:

$$x_i = x_j \Leftrightarrow S_x(i) = j, \text{ for } i \in S \text{ and } j \in \{0, 1\}.$$

**Example 3.2.** Let  $S = \{0, 1, 2, 3, 4\}$  be a set with a partial order over  $S$  showed in the Figure 1(a).

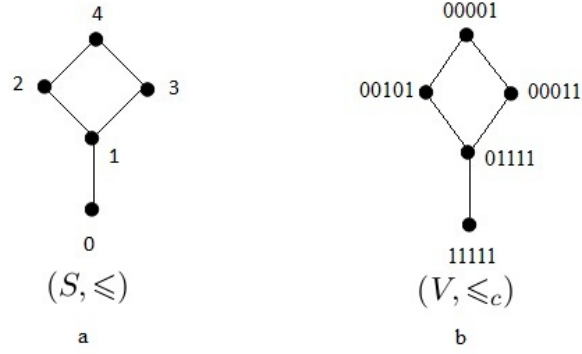


Figure 1: a)partial ordering. b)order relation  $\leq_c$

then

$S_s$	0	1	2	3	4
$S_0$	1	1	1	1	1
$S_1$	0	1	1	1	1
$S_2$	0	0	1	0	1
$S_3$	0	0	0	1	1
$S_4$	0	0	0	0	1

and thus  $V1 - P = \{11111, 01111, 00101, 00011, 00001\}$ .

**Example 3.3.** Let  $S = \{0, 1, 2, 3, 4\}$  be a set with a partial order over  $S$  showed in the figure 2(a).

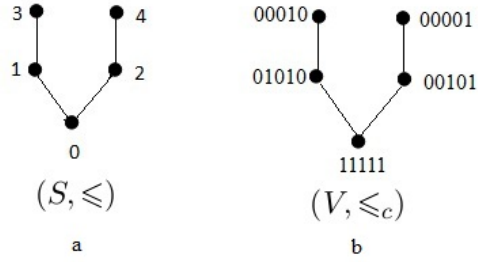


Figure 2: a)partial ordering. b)order relation  $\leq_c$

then

$S_s$	0	1	2	3	4
$S_0$	1	1	1	1	1
$S_1$	0	1	0	1	0
$S_2$	0	0	1	0	1
$S_3$	0	0	0	1	0
$S_4$	0	0	0	0	1

and thus  $V2 - P = \{11111, 01010, 00101, 00010, 00001\}$ .

**Example 3.4.** Let  $S = \{A, B, C, D\}$  be a set with a partial order over  $S$  as in the Figure 3(a).

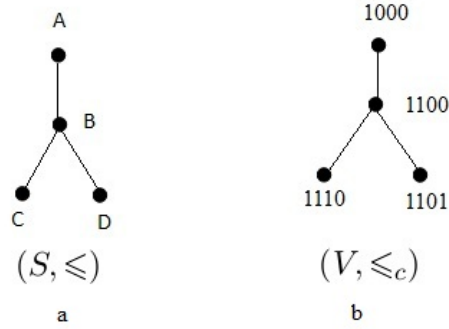


Figure 3: a)partial ordering. b)order relation  $\leq_c$

then

$S_s$	A	B	C	D
$S_A$	1	0	0	0
$S_B$	1	1	0	0
$S_C$	1	1	1	0
$S_D$	1	1	0	1

therefore  $V3 - P = \{1000, 1100, 1110, 1101\}$ .

In the following, we will compute binary block-code based on Definition 2.4. for  $BCK$ -algebras. We will show that there is a correspondence between the ordered relation on  $BCK$ -algebra and partial ordered set.

**Example 3.5.** Let  $X = \{0, 1, 2, 3, 4\}$  be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	4	0

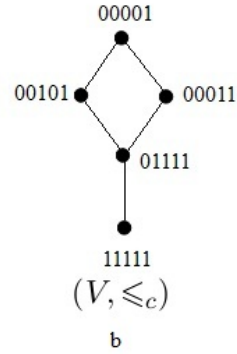
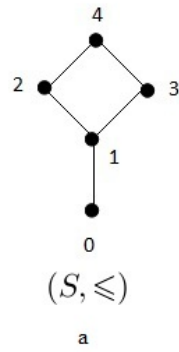


Figure 4: a)ordered relation. b)order relation  $\leq_c$

The above figure is the ordered relation on  $X$ .

Let  $\tilde{A} : X \rightarrow X$  be a BCK-function on  $X$  given by

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

then

$\tilde{A}_x$	0	1	2	3	4
$\tilde{A}_0$	1	1	1	1	1
$\tilde{A}_1$	0	1	1	1	1
$\tilde{A}_2$	0	0	1	0	1
$\tilde{A}_3$	0	0	0	1	1
$\tilde{A}_4$	0	0	0	0	1

thus  $V1 - B = \{11111, 01111, 00101, 00011, 00001\}$ .

**Example 3.6.** Let  $X = \{0, 1, 2, 3, 4\}$  be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	2	4	0

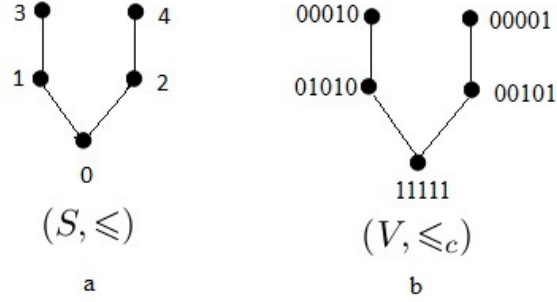


Figure 5: a)ordered relation. b)order relation  $\leq_c$

The above figure is the ordered relation on  $X$ .

Let  $\tilde{A} : X \rightarrow X$  be a BCK-function on  $X$  given by

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

then

$\tilde{A}_x$	0	1	2	3	4
$\tilde{A}_0$	1	1	1	1	1
$\tilde{A}_1$	0	1	0	1	0
$\tilde{A}_2$	0	0	1	0	1
$\tilde{A}_3$	0	0	0	1	0
$\tilde{A}_4$	0	0	0	0	1

thus  $V2 - B = \{11111, 01010, 00101, 00010, 00001\}$ .

**Remark 3.7.** On a partial ordered set with a minimum element  $\theta$  we can define a *BCK*-algebra structure(see [Fl; 14], (2.1)) From the obtained block-codes by the aforesaid methods, it is obvious that  $V1 - P = V1 - B$  and  $V2 - P = V2 - B$ . We think that the problem occurred because we use only the order of *BCK*-algebra, not its algebraic properties. From above examples, it is obvious that the method presented in paper [Ju, So; 11] dose not depend on algebraic properties of *BCK*-algebra. Also the obtained codes are not good codes, since their Hamming distance is not good. According to the figures 1 to 5, there is a one-to-one correspondence between the ordering relation  $\leq$  and order relation  $\leq_c$ .

Let  $X$  be a *BCK*-algebra and  $V$  be a linear binary block-code with  $n$  codewords of length  $n$ . We consider the matrix  $M_V = (m_{i,j})_{i,j \in \{1,2,\dots,n\}} \in \mathcal{M}_n(\{0,1\})$  with the rows consisting of the codewords of  $V$ . This matrix is called *the matrix associated to the code V*. We consider the codewords in  $V$  lexicographic ordered in the ascending sense. With this remark, for  $V = \{w_1, \dots, w_n\}$ , we denote lines in  $M_V$  with  $L_{w_1}, \dots, L_{w_n}$ . Obviously,  $w_1 = \underbrace{00\dots0}_{n\text{-time}}$ . On  $V$ , we define the following multiplication " $*$ "

$$w_i * w_j = w_k \text{ if and only if } L_{w_i} + L_{w_j} = L_{w_k}. \quad (2.1.)$$

**Proposition 3.8.** *With this multiplication,  $(V, *, \theta)$ , where  $\theta = w_1$ , becomes an abelian group.*  $\square$

**Remark 3.9.** The above group is a *BCI*-algebra.



**Example 3.10.** We consider the binary linear code  $C = \{0000, 0001, 0010, 0011\} = \{\theta, A, B, C\}$ . The associated BCI-algebra(group) is  $X = \{\theta, A, B, C\}$  with zero element  $\theta$  and multiplication given in the following table:

$*$	$\theta$	$A$	$B$	$C$
$\theta$	$\theta$	$A$	$B$	$C$
$A$	$A$	$\theta$	$C$	$B$
$B$	$B$	$C$	$\theta$	$A$
$C$	$C$	$B$	$A$	$\theta$

**Definition 3.11.** Let  $(X, *, \theta)$  be a BCI/BCK-algebra, and  $I \subseteq X$ . We say that  $I$  is a *right-ideal* if  $\theta \in I$  and  $x \in I, y \in X$  imply  $x * y \in I$ . An ideal  $I$  of a BCI/BCK-algebra  $X$  is called a *closed ideal* if it is also a *subalgebra* of  $X$  (i.e.  $\theta \in I$  and if  $x, y \in I$  it results that  $x * y \in I$ ).

Let  $C$  be a binary block code. In Theorem 2.9, from [Fl; 14], we find a BCK-algebra  $X$  such that the obtained binary block-code  $V_X$  contains the binary block-code  $C$  as a subset.

Let  $C$  be a binary block code with  $m$  codewords of length  $q$ . With the above notations, let  $X$  be the associated BCK-algebra and  $W = \{\theta, w_1, \dots, w_{m+q}\}$  the associated binary block code which include the code  $C$ . We consider the codewords  $\theta, w_1, w_2, \dots, w_{m+q}$  lexicographic ordered,  $\theta \geq_{lex} w_1 \geq_{lex} w_2 \geq_{lex} \dots \geq_{lex} w_{m+q}$ . Let  $M \in \mathcal{M}_{m+q+1}(\{0, 1\})$  be the associated matrix with the rows  $\theta, w_1, \dots, w_{m+q}$ , in this order. We denote with  $L_{w_i}$  and  $C_{w_j}$  the lines and columns in the matrix  $M$ . The sub-matrix  $M'$  of the matrix  $M$  with the rows  $L_{w_1}, \dots, L_{w_m}$  and the columns  $C_{w_{m+1}}, \dots, C_{w_{m+q}}$  is the matrix associated to the code  $C$ .

**Proposition 3.12.** With the above notations, we have that  $\{\theta, w_{m+1}, \dots, w_{m+q}\}$  determines a closed right ideal in the algebra  $X$ .

*Proof.* Let  $Y = \{\theta, w_{m+1}, \dots, w_{m+q}\}$ . Due to the multiplications and the order relation  $\preceq$  given by the relations (2.1) and (1.1) from [Fl; 14], we can have only the following two possibilities:  $w_i * w_j = \theta$  or  $w_i * w_j = w_i$ . Therefore  $Y$  is a

right-ideal in  $X$ . The multiplication (2.1.) is :

$$\begin{cases} \theta * x = \theta \text{ and } x * x = \theta, \forall x \in X; \\ x * y = \theta, \text{ if } x \leq y, \quad x, y \in X; \\ x * y = x, \text{ otherwise.} \end{cases}$$

□

□

**Remark 3.13.** From Proposition 3.12, we obtain that to each binary block code we can associate a  $BCK$ -algebra in which this code determines a right ideal.

Let  $A$  be a nonempty set and  $X$  be a  $BCK$ -algebra.

**Proposition 3.14.** *Let  $C$  be a binary block code with  $m$  codewords of length  $q$  and let  $X$  be the associated  $BCK$ -algebra, as the above. Therefore, there are the sets  $A$  and  $B \subseteq X$ , the  $BCK$ -function  $f : A \rightarrow X$  and a cut function  $f_r$  such that*

$$C = \{f_r : A \rightarrow \{0, 1\} / f_r(x) = 1, \text{ if and only if } r * f(x) = \theta, \forall x \in A, r \in B\}. \square$$

**Remark 3.15.** i) Let  $S = \{1, 2, \dots, n\}$  be the set with  $n$  elements. We know that  $(\mathcal{P}(S), \Delta, \cap)$  is a Boolean ring, where  $\mathcal{P}(S)$  is the power set of the set  $S$ ,  $\Delta$  is symmetric difference of the sets and  $\cap$  is the intersection of two sets. Let  $\mathfrak{F} = \{f : S \rightarrow \{0, 1\} / f \text{ function}\}$ . To each  $f \in \mathfrak{F}$  corresponds a binary block codeword. To each binary block codeword  $c_1$  corresponds an element from  $\mathcal{P}(S)$ . Indeed, to each binary codeword  $c = (i_1, \dots, i_n)$  we will associate the set  $I_c = \{j_1, j_2, \dots, j_k\} \in \mathcal{P}(S)$  such that  $i_{j_1} = i_{j_2} = \dots = i_{j_k} = 1$ .

ii) Using the above established correspondence, if  $C = \{c_1, c_2, \dots, c_m\}$  is a linear binary block code and  $Q = \{I_{c_1}, I_{c_2}, \dots, I_{c_m}\} \subseteq \mathcal{P}(S)$ , where  $I_{c_i}$  is the associated subset for the codeword  $c_1$ , then  $Q$  is a sub-ring in the Boolean ring  $(\mathcal{P}(S), \Delta, \cap)$ . It results a bijective map between the sub-rings of the Boolean ring  $(\mathcal{P}(S), \Delta, \cap)$  and linear binary block codes with codewords of length  $n$ .

**Example 3.16.** i) Let  $C = \{0000, 0001, 0010, 0011\} = \{w_6, w_7, w_8, w_9\}$  be a linear binary block code and let  $X = \{\theta, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9\}$  be the

obtained BCK-algebra as in Theorem 2.9 from [Fl; 14]. The multiplication of this algebra is given in the below table

*	$\theta$	$w_2$	$w_3$	$w_4$	$w_5$	$w_6$	$w_7$	$w_8$	$w_9$
$\theta$	$\theta$	$\theta$	$\theta$	$\theta$	$\theta$	$\theta$	$\theta$	$\theta$	$\theta$
$w_2$	$w_2$	$\theta$	$w_2$	$w_2$	$w_2$	$w_2$	$w_2$	$\theta$	$\theta$
$w_3$	$w_3$	$w_3$	$\theta$	$w_3$	$w_3$	$w_3$	$w_3$	$\theta$	$w_3$
$w_4$	$w_4$	$w_4$	$w_4$	$\theta$	$w_4$	$w_4$	$w_4$	$w_4$	$\theta$
$w_5$	$w_5$	$w_5$	$w_5$	$w_5$	$\theta$	$w_5$	$w_5$	$w_5$	$w_5$
$w_6$	$w_6$	$w_6$	$w_6$	$w_6$	$w_6$	$\theta$	$w_6$	$w_6$	$w_6$
$w_7$	$w_7$	$w_7$	$w_7$	$w_7$	$w_7$	$w_7$	$\theta$	$w_7$	$w_7$
$w_8$	$w_8$	$w_8$	$w_8$	$w_8$	$w_8$	$w_8$	$w_8$	$\theta$	$w_8$
$w_9$	$w_9$	$w_9$	$w_9$	$w_9$	$w_9$	$w_9$	$w_9$	$w_9$	$\theta$

From Proposition 3.12, we remark that  $\{\theta, w_6, w_7, w_8, w_9\}$  is a right ideal in the BCK-algebra  $X$ . From Proposition 3.14, for  $A = \{w_6, w_7, w_8, w_9\}$  and  $B = \{w_2, w_3, w_4, w_5\}$ , we recover the initial code  $C$ .

**Example 3.17.** For the same linear binary block code  $C = \{0000, 0001, 0010, 0011\}$ , let  $Q = \{\emptyset, \{4\}, \{3\}, \{3, 4\}\}$  as in Remark 3.15 ii). It is clear that  $Q$  is a sub-ring in the Boolean ring  $(\mathcal{P}(\{1, 2, 3, 4\}), \Delta, \cap)$  and  $C$  can be considered as a sub-ring of this Boolean ring.

**Remark 3.18.** In [Fl; 14], Theorem 2.2, the studied binary block codes have Hamming distance equal with 1. In the same paper, Theorem 2.9, to an arbitrary binary block code  $C$  we associate a BCK algebra  $X$  and the code associated to this algebra includes the code  $C$ . Proposition 3.14 improved this theorem since we can even obtain the code  $C$  and from Proposition 3.12 we have that the code  $C$  generate a right ideal in the algebra  $X$ .

**Remark 3.19.** The obtained results of above remarks and propositions can be illustrated by partially ordered sets. Let  $C$  be a binary block code with  $m$  codewords of length  $q$ . According to Proposition 2.8 and Theorem 2.9 in [Fl; 14], we can find the matrix  $M \in \mathcal{M}_{m+q+1}(\{0, 1\})$  that is the matrix associated to the code  $C$ . Let  $S$  be the associated partially ordered set. Therefore, there are the sets  $A$  and  $B \subseteq S$  and the function  $f : A \rightarrow S$ , such that we can define the bellow set:

$$C = \{f_r : A \rightarrow \{0, 1\} \mid f_r(b) = 1, \text{ if and only if } r \leq b, \forall b \in A, r \in B\}.$$

Here,  $A = \{m + 2, \dots, m + q + 1\}$  and  $B = \{2, \dots, m + 1\}$ .

**Example 3.20.** Let  $C = \{0000, 0001, 0010, 0011\}$  be a linear binary block code and let  $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ . In this example  $m = q = 4$ . The matrix associated to the code  $C$  is:

1	1	1	1	1	1	1	1	1
0	1	0	0	0	0	0	1	1
0	0	1	0	0	0	0	1	0
0	0	0	1	0	0	0	0	1
0	0	0	0	1	0	0	0	0
0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	1

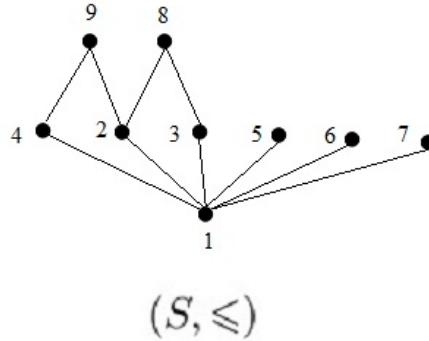


Figure 6: partial ordering.

The above figure is partial ordering over  $S$ . From above Proposition,  $A = \{6, 7, 8, 9\}$  and  $B = \{2, 3, 4, 5\}$  that from  $A$  and  $B$ , we can recover the initial code  $C$ .

**Conclusions.** Even if, from the above examples, appears that the associated binary block codes depend only from the order relation defined on a  $BCK$ -algebra, will be very interesting to study in a further paper how and if the properties of  $BCK$ -algebras can influence the properties of the associated binary block codes.

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